

ABSENCE OF PARTICLE PRODUCTION AND FACTORIZATION
OF THE S-MATRIX IN 1+1 DIMENSIONAL MODELS*

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ABSTRACT

In massive, 1+1 dimensional, local, quantum field theories the existence of two conserved charges is shown to be a sufficient condition for the absence of particle production and factorization of the S-matrix. These charges must commute and be integrals of local current densities. Their transformation properties under the Lorentz group must be different and different from a vector or a scalar. Also, they must not annihilate any single particle momentum eigenstate.

* Research supported in part by the National Science Foundation under Grant No. PHY77-22864.

I. Introduction

Recently, substantial progress has been made in 1+1 dimensional S-matrix theory – the exact on-mass shell S-matrix has been calculated for a number of models, including the sine-Gordon equation, non-linear σ model, and the multi-fermion Gross–Neveu models. This work was pioneered by the authors Zamolodchikov, Zamolodchikov, Karowski, Thun, Truong, and Weiss [1–4]. The S-matrices of those models, although highly non-trivial, have two important properties which are essential to their calculability: first, there is an absence of particle production, that is the momentum set of the outgoing particles is identical to the momentum set of the incoming particles; second, the N particle S-matrix factorizes into a product of two particle S-matrices, regardless of the impact parameters of the incoming particles. These properties, together with crossing, unitarity and analyticity, enable us to write solvable, non-linear equations for the two-particle scattering amplitudes. The general N particle scattering amplitudes can then be constructed as the appropriate products of the two-particle scattering amplitudes.

The absence of particle production and factorization properties of the S-matrix are implied by the existence in the quantum theory of an infinite number of conserved charges [5–7]. These charges are integrals of local current densities which transform under the Lorentz group according to higher and higher rank representations. In quantum perturbation theory, the demonstration that a given model has an infinite number of conserved charges is difficult and has only been achieved in a few models [8]. However, the first few charges of lowest rank are easily shown to be conserved. Therefore, it is important to determine whether an infinite number of conserved charges is a necessary condition to imply the properties of absence of particle

production and factorization of the S-matrix.

Indications from quantum mechanics [9] and a special case in quantum field theory [9,10] suggest that the infinite number of conserved charges are not all necessary. This suggestion should not be surprising, since many years ago Coleman and Mandula [11] proved a theorem which tells us that in more than one spatial dimension a theory possessing a conserved charge transforming under the Lorentz group as a tensor of second or higher rank necessarily has a trivial S-matrix.

In this paper, I show that two conserved charges are sufficient to deduce the two essential properties of the S-matrix that allow it to be calculated, given only massive particles in the particle spectrum and standard analyticity assumptions of the S-matrix. The charges must transform under the Lorentz group differently from each other, differently than a vector or a scalar, and be integrals of local current densities. The important components of the proof are the transformations generated by the extra conserved charges on localized single particle states and the macroscopic causal properties of space-time.

The structure of the paper is as follows. In section II, I describe the extra conserved charges and their properties. In section III, the transformations generated by the extra charges on localized particle states are discussed. Section IV addresses the properties of the two-particle scattering amplitude. Section V discusses the properties of the three or more particle scattering amplitudes. Section VI contains my conclusions. Two appendices are attached which contain proofs of technical results required in the argument.

II. The Extra Conserved Charges

The conserved quantities that we ordinarily encounter in 1+1 dimensional quantum field theories are the momentum and internal symmetry charges. The momentum, p_μ , transforms like a vector under the Lorentz group while the internal symmetry charges, I_i , transform as scalars.

The extra conserved charges that are required to impose the properties of absence of particle production and factorization of the S-matrix transform under the Lorentz group like tensors of second and/or higher rank. I use light-cone co-ordinates to describe the Lorentz transformation properties of these charges more precisely. Defining $x^\pm = x^0 \pm x^1$, then, under the Lorentz group,

$$x^\pm \rightarrow \Lambda^\pm x^\pm$$

where Λ is equal to the exponential of the change in the rapidity of a massive particle. I require that the extra conserved charges, defined as Q^+ and Q^- , transform under a Lorentz transformation as follows

$$Q^+ \rightarrow \Lambda^{+m} Q^+ \quad (2.1a)$$

$$Q^- \rightarrow \Lambda^{-n} Q^- \quad (2.1b)$$

where the integers m and n satisfy the restrictions $m \geq n > 1$. The left inequality defines the positive space direction. These constraints on m and n guarantee that these extra charges transform under the Lorentz group differently from each other, and differently than a vector or a scalar. The extra charges are labeled this way because in parity conserving theories for every Q^+ there also exists the parity conjugate Q^- for which $n=m$; however the conservation of parity is not used in the proof.

The two charges, Q^\pm , are assumed to be spatial integrals of the time component of some local, conserved current density. Thus, Q^\pm are conserved and commute with the generator of space-time translations, the momentum

operator, $[Q^\pm, p_\mu] = 0$. Therefore, single particle momentum eigenstates, $|p\rangle$, can be found which are also eigenstates of Q^+ and Q^- .

In general, the particle spectrum consists of a number of mass multiplets with the particles in each multiplet distinguished by the internal symmetry charges. Since the mass operator, M^2 , is the square of the momentum operator, $p_\mu p^\mu$, it also commutes with Q^\pm . Therefore, the single particle momentum eigenstates, which are also eigenstates of Q^\pm , are linear combinations of particles within the same mass multiplet.

There are two requirements for the action of Q^\pm on all multiplets that must be satisfied. First, I require that Q^+ and Q^- do not annihilate any linear combination of particles in any multiplet. Second, Q^+ and Q^- must have simultaneous eigenstates; this condition is satisfied if $[Q^+, Q^-] = 0$. As a result of these requirements, in every multiplet there exists a set of single particle momentum eigenstates, $|p_a\rangle$, which are also eigenstates of Q^\pm . The eigenvalues are determined by Lorentz invariance, therefore

$$Q^+ |p_a\rangle = n_a^+ (p^+)^m |p_a\rangle \quad (2.2a)$$

$$Q^- |p_a\rangle = n_a^- (p^-)^n |p_a\rangle \quad (2.2b)$$

where n_a^\pm are non-zero Lorentz scalars that depend on the particular linear combination of particle states within a multiplet.

If Q^\pm commute with all internal symmetry charges, $[Q^\pm, I_i] = 0$ for all i , then $n_a^+ = n_b^+$ for all a, b in the same multiplet but the n 's may differ from multiplet to multiplet. Also if parity is conserved and Q^\pm are parity conjugates then $n_a^+ = n_a^-$ for all eigenstates of Q^\pm .

In the remainder of this paper, I treat the eigenstates of Q^\pm as being the particle states in the theory. Since the transformation between the eigenstates of Q^\pm and the eigenstates of I_i is invertible, if the S-matrix is known in one of these bases, it is known in the other basis.

Since Q^{\pm} are integrals of local current densities, their action on widely separated, multi-particle states is the sum of the actions on the individual particles. Therefore,

$$Q^{\pm}|p_1 \dots p_k\rangle = \left[\sum_{i=1}^k n_i^{\pm}(p_i^{\pm})^m \right] |p_1 \dots p_k\rangle \quad (2.3)$$

and similarly for Q^- . Thus in a scattering process as well as the asymptotic conservation of Σp_i^+ and Σp_i^- , these models also have the asymptotic conservation of $\sum n_i^+(p_i^+)^m$ and $\sum n_i^-(p_i^-)^n$.

Throughout the paper, I frequently use different linear combinations of Q^+ and Q^- . Therefore, it is useful to introduce the conserved charge

$$Q_{\theta} \equiv (\cos\theta)Q^+/m - (\sin\theta)Q^-/n \quad (2.4)$$

where $\theta \in [0, 2\pi]$. Q_{θ} has the same properties as Q^{\pm} except for trivial changes in its eigenvalues with the eigenstates, $|p_a\rangle$.

In the next section, the action of Q_{θ} on localized single particle states will be determined.

III. Localized States

In a theory with a conserved charge Q_0 , a general scattering amplitude

$$\langle p_1 \dots p_k | S | q_1 \dots q_\ell \rangle$$

$$\text{is equal to} \quad \langle p_1 \dots p_k | e^{-iaQ_0} S e^{iaQ_0} | q_1 \dots q_\ell \rangle$$

for any real a . However, if the incoming and outgoing particles are localized in space and the transformations generated by Q_0 on these localized states move the particles by amounts which depend on their type and momentum, then the causal structure of spacetime can be used to put restrictions on the scattering amplitudes. It is important, at this point, to understand the action of $\exp[iaQ_0]$ on a localized single particle state for all real a .

But because $Q_0 \pm \pi = -Q_0$ only positive a need be considered with θ in the interval $[0, 2\pi]$.

A localized single particle state with mean momentum (\bar{E}, \bar{p}) will be represented by the wavefunction

$$\Psi(x, t) = N \int dp e^{i\bar{p}x - Et} f(p) \quad (3.1a)$$

where N is a normalization factor,

$$f(p) = -(p - \bar{p})^2 / 2(\bar{E} \delta\phi)^2 + i(p(x - x_0) - E(t - t_0)) , \quad (3.1b)$$

$$E = (p^2 + \mu^2)^{1/2} = \gamma\mu ,$$

μ being the mass of the particle and (t_0, x_0) the co-ordinates of the minimum spread of the wavepacket.

The physical interpretation of $\delta\phi$ is that it is half the velocity spread of the wavepacket when viewed from the frame in which $\bar{p} = 0$. Therefore $\delta\phi$ should satisfy the following constraints,

$$0 < \delta\phi \ll 1 . \quad (3.2)$$

The right-hand constraint insures that the minimum spread is much larger than the Compton wavelength in this frame and therefore avoids the problems associated with localizing a single relativistic particle in too small a region.

At any particular time, the region of space where the particle is most likely to be found is also of interest. To find this region, a stationary phase analysis is performed on $f(p)$. This analysis shows that the center of the wavepacket at time t , is $x(t)$ given by

$$x(t) = x_0 + \bar{v}(t-t_0) \quad (3.3)$$

where $\bar{v} \equiv \bar{p}/\bar{E}$. The extent of the wavepacket about $\bar{x}(t)$ is defined as $\delta x(t)$

$$\delta x(t) = \kappa |t-t_0| (\delta\phi/\bar{\gamma}^2) \quad (3.4)$$

for large $|t-t_0|$. For small $|t-t_0|$, $\delta x(t)$ is a constant whose value will be of no concern. $(\delta\phi/\bar{\gamma}^2)$ is half the velocity spread of the wavepacket as measured in the frame in which the mean velocity is \bar{v} . The quantity κ is chosen such that the probability of finding the particle outside the interval $(\bar{x}-\delta x, \bar{x}+\delta x)$ is extremely small. There is no point in discussing in detail the criteria for choosing κ ; as will become clear later, my arguments are valid for any non-zero κ .

Having chosen the momentum and the co-ordinates of minimum spread, the only freedom that is left in the wavepacket is the value of $\delta\phi$ satisfying the constraint (3.2).

The action of $\exp[i\alpha Q_0]$ on such a localized single particle state replaces $f(p)$ in the wavefunction of Eq. (3.1) by

$$\tilde{f}(p) = f(p) + i\alpha(\cos\theta n^+ p^m/m - \sin\theta n^- p^{-n}/n) \quad (3.5)$$

Again, stationary phase analysis is used to find the region of space where the particle is most likely to be found. The result is the same as Eqs.

(3.3) and (3.4) but with (t_0, x_0) replaced by (\bar{t}_0, \bar{x}_0) where

$$\bar{t}_0 = t_0 + \alpha \left[n^+ (\bar{p}^+)^{m-1} \left(\frac{m-\bar{v}}{1-\bar{v}} \right) \cos\theta - n^- (\bar{p}^-)^{n-1} \left(\frac{n-\bar{v}}{1+\bar{v}} \right) \sin\theta \right] \quad (3.6a)$$

$$\bar{x}_0 = x_0 + \alpha \left[n^+ (\bar{p}^+)^{m-1} \left(\frac{m-\bar{v}}{1-\bar{v}} \right) \cos\theta - n^- (\bar{p}^-)^{n-1} \left(\frac{n+\bar{v}}{1+\bar{v}} \right) \sin\theta \right] \quad (3.6b)$$

As a simple check, it is worth noting that if $Q^\pm = p^\pm$ then $n^+ = n^- = 1$ and $n^+ = n^- = 1$ for all particles and (3.6) gives the result of a spacetime translation as expected.

The shift of the center of the wavepacket at constant time equals

$$(\bar{x}_0 - x_0) - \bar{v}(\bar{t}_0 - t_0) = -\alpha \left[n^+ (\bar{p}^+)^m \cos\theta + n^- (\bar{p}^-)^n \sin\theta \right] / \bar{E} \quad (3.7)$$

A general N particle scattering process can be represented by the impinging of N such wavepackets. For the moment, consider two such wavepackets which do not interact with one another. Then the volume of spacetime where these two wavepackets overlap is the possible region of interaction. The co-ordinates of the center of this region are the co-ordinates of the intersection of the centers of the wavepackets. The edges of this region are obtained from the intersection of the extent of the wavepackets.

It is more convenient to write the expressions for these co-ordinates in terms of the rapidities of the particles. Thus, describing the particles in terms of their rapidities, $\bar{p}_i = \mu_i \sinh \phi_i$, the intersection co-ordinates of the centers of the i^{th} and j^{th} wavepacket, (t_{ij}, r_{ij}) , are given by

$$\begin{bmatrix} t_{ij} \\ r_{ij} \end{bmatrix} = \alpha \begin{bmatrix} a_{ij}, b_{ij} \\ c_{ij}, d_{ij} \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + O(1) \quad (3.8a)$$

which will be written as $x_{ij} = \alpha M_{ij} \hat{\phi}_0 + O(1)$

with the obvious identifications. The coefficients of the intersection matrix, M_{ij} , are given by

$$a_{ij} \pm c_{ij} = (\hat{n}_i^+ e^{m\phi_i \pm \phi_j} - \hat{n}_j^+ e^{m\phi_j \pm \phi_i}) / \sinh(\phi_i - \phi_j) \quad (3.8b)$$

$$b_{ij} \pm d_{ij} = (\hat{n}_i^- e^{-m\phi_i \pm \phi_j} - \hat{n}_j^- e^{-m\phi_j \pm \phi_i}) / \sinh(\phi_i - \phi_j) \quad (3.8c)$$

where $\hat{n}_i^\pm = n_i^\pm u_i^{n-1}$ and $\hat{n}_i^- = n_i^- v_i^{n-1}$. The terms $O(1)$ contain the initial condition but since we will be always interested in letting a become large these are of no interest.

To obtain the edges of the overlap region, it is sufficient to let $\phi_{i,j} \rightarrow \phi_{i,j} \pm \kappa \delta\phi$ in the intersection matrix M_{ij} . For the overlap region of the two wavepackets to be small, the change in M_{ij} , δM_{ij} must be small. That is

$$\|\delta M_{ij}\| / \|M_{ij}\| \ll 1. \quad (3.9)$$

If $\delta\phi$ is such that $|\phi_i - \phi_j| \gg 2\kappa\delta\phi$ and $1 \gg m\kappa\delta\phi$ then δM_{ij} is proportional to $\delta\phi$ so that by letting $\delta\phi$ become small enough, the inequality (3.9) can be satisfied. $|\phi_i - \phi_j| > 2\kappa\delta\phi$ has a simple physical interpretation, that is, if it is not satisfied then the wavepackets i and j do not separate at large positive and/or negative times.

The intersection matrix, M_{ij} , contains all the required information about the region of overlap of the i th and j th wavepackets. This matrix has a number of useful properties.

First given three particles i , j and k such that $M_{ij} = M_{ik}$ then $M_{jk} = M_{ij} = M_{ik}$ provided $\phi_j \neq \phi_k$. This transitivity property of M_{ij} is obvious, since if, for all 0 , i intersects both j and k at the same spacetime point then so must j and k intersect at that point. If $\phi_j = \phi_k$, M_{jk} is not defined

unless $\hat{n}_j^\pm = \hat{n}_k^\pm$. M_{ij} is the matrix that gives the co-ordinates of the minimum spread of the wavepacket after a transformation generated by \hat{Q}_0 when used in Eq. (3.8).

The second property is more important and will be called the isolated points lemma with the proof given in Appendix A.

Lemma: Given three particles i , j and k with their type specified and with at least one particle, say i , with rapidity different than the others, then the condition

$$N_{ij} = N_{ik} \quad (3.10)$$

can only occur at isolated points in $\phi_j - \phi_i$ versus $\phi_k - \phi_i$ rapidity difference space unless particle j has the same \hat{n}_i^\pm as particle k and $\phi_j = \phi_k$. If all particles have the same rapidity then this is an isolated point in rapidity difference space.

These properties of M_{ij} will be used in the proof that the S-matrix has an absence of particle production and factorizes.

IV. Two-particle Scattering

This section addresses the restrictions placed on the scattering of two particles into N particles, by the existence of the extra conserved charges. The general two-particle scattering amplitude is

$$\langle \phi_3, \dots, \phi_{N+2} | S | \phi_1, \phi_2 \rangle$$

where the ϕ_i 's are the rapidities of the particles and are ordered such that $\phi_1 > \phi_2$ and $\phi_3 > \phi_4 > \dots > \phi_{N+2}$.

If the incoming and outgoing particles are localized in space by placing them in wavepackets described by Eq. (3.1), then the spacetime diagram of the process is as shown in Fig. 1. Each one of the lines in Fig. 1 represents the center of the wavepacket of a particle either before any collision for the incoming particles or after all collisions for the outgoing particles.

The macroscopic causality properties of spacetime impose restrictions on the intersection co-ordinates of these lines obtained by extrapolating them forward (backward) in time for the incoming (outgoing) particles. The relevant restriction for this proof is an inequality that relates the collision time of the two incoming particles, 1 and 2, and the intersection time of the slower incoming particle, 2, and the fastest outgoing particle, 3, of particles 1 and 2 and undergo no subsequent interactions.

$$t_{21} \leq t_{23}. \quad (4.2)$$

A violation of this inequality is allowed on microscopic but not macroscopic time scales. However, this proof deals with only macroscopic time scales. Therefore this violation is of no consequence.

The inequality (4.2) can be demonstrated by following the time evolution of the particle which is always furthest to the right after the collision of the incoming particles. This particle either eventually emerges as the asymptotic particle which is the fastest, that is particle 3, or is involved

in a subsequent collision or decays. However, even after a collision or decay the particle which is now furthest to the right is always moving faster than the particle previously furthest to the right. Therefore, when projected backwards in time this new line intersects the particle 2 line

$$(4.1) \quad \text{after the collision of particle 1. Since the particle furthest to the right}$$

eventually is the fastest asymptotic particle, the inequality (4.2) holds.

If I now use the freedom to make transformations generated by Q_θ , the

amplitude (4.1) is equal to

$$\langle \phi_3, \dots, \phi_{N+2} | e^{-i\alpha Q_\theta} S e^{+i\alpha Q_\theta} | \phi_1, \phi_2 \rangle. \quad (4.3)$$

After this transformation, the intersection times of the center of the wavepackets are transformed according to Eq. (3.8). Therefore,

$$t_{23}-t_{21} = \alpha[(a_{23}-a_{21})\cos\theta + (b_{23}-b_{21})\sin\theta] + O(1). \quad (4.4)$$

If $a_{23} \neq a_{21}$ or $b_{23} \neq b_{21}$ then by an appropriate choice of θ and by permitting α to become large, $t_{23}-t_{21}$ can be made macroscopically large and negative. This violates the inequality (4.2). Therefore, $a_{23} = a_{21}$ and $b_{23} = b_{21}$, that is $t_{23} = t_{21}$ for all θ . This implies that $r_{23} = r_{21}$ for all θ . Hence, $M_{23} = M_{21}$ and particle 3 must come directly from the collision region

Applying this argument to the next to fastest particle, since it does

not interact with the fastest particle after the initial collision, the same result is obtained. This argument can be applied to all particles in the outstate, which have different rapidity than particle 2, by cascading from the fastest to the slowest. The argument can also be applied to the slowest particle with trivial modifications, thus taking care of the particles in the outstate which have rapidities less than or equal to the rapidity of particle 2.

Therefore, in a two-particle collision all pairs of particles, with unequal rapidities, must have the same intersection matrix, M_{ij} , to be consistent

with the transformations generated by Q_0 and the macroscopic causal structure of spacetime.

The maximum number of particles that can have the same N_{ij} for all pairs is three times the number of different ratios (\hat{n}^+/\hat{n}^-) that occur in the particle spectrum. The proof of this statement is presented in Appendix B. This statement is not sufficient to eliminate particle production in a two-particle collision.

However, if I consider one outgoing particle, k , in a two-particle scattering process, then one or both of the following is true:

$$(i) \quad \phi_k \neq \phi_1 \quad \text{and} \quad N_{1k} = N_{12},$$

$$(ii) \quad \phi_k \neq \phi_2 \quad \text{and} \quad N_{2k} = N_{21}.$$

From the isolated points Lemma stated at the end of section III, we know that by varying the rapidity difference of the incoming particles, $\phi_2 - \phi_1$, by a small amount, that no small change in ϕ_k can maintain either (i) or (ii) above unless for (i) $\phi_k = \phi_2$ and $\hat{n}_k^\pm = \hat{n}_2^\pm$ or for (ii) $\phi_k = \phi_1$ and $\hat{n}_k^\pm = \hat{n}_1^\pm$. Therefore, unless an outgoing particle has the same rapidity and \hat{n}^\pm as an incoming particle, it can only be produced at isolated points in the rapidity difference of the incoming particles. An S-matrix which produces a particle at such an isolated point in the rapidity difference of the incoming particles does not have the correct analyticity and continuity properties of an S-matrix of a field theory with local interactions. Therefore, such isolated production does not exist in the models under discussion.

At most, there are two particles in the outstate, with the same rapidities and \hat{n}^\pm as those in the instate. Energy-momentum conservation implies that there are two particles in the outstate and they must have the same momentum and hence mass as the particles in the instate, that is

$$\langle p_1, \dots, p_N | S | q_1, q_2 \rangle \propto \delta_{N2}^{(2)} (p_1 - q_1) \delta^{(2)} (p_2 - q_2), \quad (4.5)$$

In this section on two-particle scattering I have shown that the final momentum set is equal to the initial momentum set. This is the absence of particle production for two-particle scattering. However, the two-particle S-matrix is non-trivial because of the possibilities of microscopic time delays or advances and, if the incoming particles have the same \hat{n}^\pm , interchanges of the internal quantum numbers.

V. Three or More Particle Scattering

In this section, I first address the scattering of three particles and then proceed by induction for the case of an arbitrary number of particles scattering. The method is the same as that used in section IV; I represent the scattering particles by localized wavepackets and perform the transformations generated by Q_0 on these states. Restrictions on the scattering amplitudes are determined by the causal structure of spacetime.

Given the rapidities, masses and n^t 's of three particles, i, j and k, and represented by wavepackets whose wavefunctions are given by Eq. (3.1); I temporarily assume that these particles do not interact. The intersection co-ordinates of the center of the wavepackets are given by Eq. (3.8).

Thus, the co-ordinate difference of two intersections is given by

$$x_{ij} - x_{ik} = \alpha(M_{ij} - M_{ik})\hat{e}_0 + O(1). \quad (5.1)$$

However, except for the case $M_{ij} = M_{ik}$, the matrix $M_{ij} - M_{ik}$ has at most one null eigenvector. By choosing \hat{e}_0 to be different than the null eigenvector, the intersection co-ordinates of the center of the wavepackets can be made arbitrarily far apart by letting α become arbitrarily large.

Allowing for the spread in the wavepackets by taking the rapidities $\phi \rightarrow \phi + \delta\phi$ and remembering that M_{ij} changes by an amount proportional to $\delta\phi$, for small $\delta\phi$, the change in the matrix $M_{ij} - M_{ik}$ is also small for sufficiently small $\delta\phi$. Therefore, the possible null eigenvector also changes only by a small amount.

Hence, by choosing $\delta\phi$ sufficiently small, \hat{e}_0 can be found such that the spacetime region of overlap of the wavepackets are separate distinct two-particle overlap regions as in Fig. 2. These two-particle overlap regions are regions of possible interaction and if I choose α to be arbitrarily large, they can be made arbitrarily far from one another.

Allowing for interactions, we know from section IV, that two particles can only scatter into two particles with the same momentum set in the out-state as the instate. This means that interactions do not change the above picture; the interactions occur as separate, distinct two-particle interactions.

Returning to the case $M_{ij} = M_{ik}$, the Lemma stated in section III is important. This Lemma states that $M_{ij} = M_{ik}$ can occur at only isolated points in the $\phi_j - \phi_i$ versus $\phi_k - \phi_i$ plane unless particles j and k have the same rapidity and n^t . If two particles have the same rapidity, this is not relevant for the three-particle S-matrix since the instate is not three distinct particles.

Hence, the three-particle S-matrix has an absence of particle production and factorizes except at these isolated points in rapidity difference space. The analyticity and continuity properties of the S-matrix for field theories with local interactions imply that this is also true at these isolated points. Therefore, the three-particle S-matrix has the desired properties.

For more than three particles in the instate, I proceed by induction. Assuming that the S-matrix factorizes and has no particle production for any number of particles less than N, I turn my attention to the N particle S-matrix.

Given the rapidities, masses and n^t 's of N particles, 1, 2, ..., N, the following condition is either true or false:

$$M_{11} = M_{1j} \quad (5.2)$$

for all i and j not equal to 1.

If this condition is not satisfied then there exist particles k and l such that $M_{1k} \neq M_{1l}$ and by following the same argument as used for three-

particle scattering there exists an \hat{c}_0 such that $x_{1k} - x_{1l} \neq 0$ and can be made arbitrarily large. This proves that the N-particle S-matrix factorizes into a product of S-matrices with less than N incoming particles. Therefore, if (5.2) is false, using the induction assumptions, the N particle S-matrix factorizes into a product of two-particle S-matrices and has an absence of particle production.

The condition (5.2) can only be true if N is less than three times the number of different ratios \hat{n}^+/\hat{n}^- in the particle spectrum and then only at isolated rapidity difference points according to the Lemma of section III. Thus, if the rapidity of say particle 2 is varied by a small amount the condition (5.2) is now false. For these isolated rapidity points the analyticity and continuity properties of the S-matrix imply that the S-matrix must have the same properties at these isolated rapidity points as at rapidity points nearby.

Therefore, in this section I have shown that the N-particle S-matrix factorizes into a product of two-particle S-matrices and has an absence of particle production.

VI. Conclusion

In massive, 1+1 dimensional, local, quantum field theories the existence of two conserved charges has been shown to be a sufficient condition for the absence of particle production and the factorization of the S-matrix. The charges required must commute, must be integrals of local current densities and transform under the Lorentz group differently, and differently than a vector or a scalar. Also, they must not annihilate any single particle momentum eigenstate,

The important ingredients for this proof are the transformations generated by these conserved charges on localized single-particle states and the macroscopic causal structure of spacetime. The question of whether the two charges are necessary to prove the above result has not been addressed.

Acknowledgements

I am grateful to Sidney Coleman for many valuable discussions during the course of this investigation. I thank the SLAC Theory Group for the hospitality extended to me during part of this investigation and the M.I.T. Computer Science Department for the use of their algebraic manipulation program NACSYA.

Appendix A

In this appendix, I prove, for positive n , the Lemma stated in section III and give a plausibility argument for negative n . The Lemma states that given three particles 1, 2 and 3, with $\phi_2 \neq \phi_1$ and $\phi_3 \neq \phi_1$, the condition

$$M_{12} = M_{13}$$

can only be satisfied at isolated rapidity difference points unless particles 2 and 3 have the same rapidity and \hat{n}_1^+ .

If $\phi_2 = \phi_3$ and $M_{12} = M_{13}$, then $\hat{n}_2^+ = \hat{n}_3^+$, and obviously particle 2 identical to particle 3 is a non-isolated solution of (A-1). Thus, I need to show that there are only isolated solutions to (A-1) with all three particles having different rapidities.

By choosing the Lorentz frame such that $\phi_1 = 0$ and the normalization of Q^+ such that $\hat{n}_1^+ = \hat{n}_1^- = 1$, the conditions under which (A-1) is satisfied are

$$\frac{(a e^{-m\phi} - e^\phi)}{\sinh \phi} = \frac{(b e^{-m\phi} - e^\phi)}{\sinh \psi} \quad (A-2a)$$

$$\frac{(c e^{-n\phi} - e^\phi)}{\sinh \phi} = \frac{(d e^{-n\phi} - e^\phi)}{\sinh \psi} \quad (A-2b)$$

where $\phi \equiv \phi_2$, $\psi \equiv \phi_3$, and $(a, b, c, d) \in (\hat{n}_2^+, \hat{n}_3^+, \hat{n}_2^-, \hat{n}_3^-)$.

I rewrite these equations as polynomial equations by substituting $x \equiv e^{-\phi}$ and $y \equiv e^{-\psi}$, and obtain

$$(y^{m-1}-b)x^{m+1} + (b-y^{m+1})x^{m-1} + a y^{m-1}(y^2-1) = 0 \quad (A-3a)$$

and for $n > 0$

$$c(1-y^2)x^{n+1} + (dy^{n+1}-1)x^n + y^2(1-dy^{n-1}) = 0 \quad (A-3b)$$

or for $n < 0$

$$(y^{|n|}-1-d)x^{|n|+1} + (d-y^{|n|+1})x^{|n|-1} + cy^{|n|-1}(y^2-1) = 0 \quad (A-3c)$$

Since polynomial equations in two variables have non-isolated solutions if common factors which depend on both x and y . Accordingly, the polynomial

and only if there are common factors, we must find the common factors of

(A-3a) and (A-3b) for positive n and of (A-3c) for negative n .

Obviously, the only common factors independent of x are $(y-1)$ and $(y+1)$, which occur when $b = d = 1$ and $(-1)^{m+1}b = (-1)^{n+1}d = 1$ respectively.

(A-1). But, $y=1$ is forbidden because it corresponds to $\phi_3 = \phi_1$, which has already been excluded. Also, $y=-1$ gives a complex rapidity for one of the par-

ticles and is thus not physically realizable. Therefore, there are no

common factors independent of x . Similarly, by writing (A-3) as polynomials in y , one can show that there are no relevant common factors in-

dependent of y .

To find common factors which depend on x and y , I use the theory of finding common zeros of two polynomials of a single variable [12]. For

positive n , consider the two polynomials in x, f, g , defined by

$$f = r x^{m+1} + s x^{m-1} + t \quad (A-4a)$$

$$g = u x^{n+1} + v x^2 + w \quad (A-4b)$$

where r through w are independent of x . Let α_i , $i = 1, \dots, (m+1)$ and β_j , $j = 1, \dots, (n+1)$ be the zeros of f and g respectively, then these two polynomi-

als have a common zero if and only if $\prod_{i=1}^{m+1} \prod_{j=1}^{n+1} (\alpha_i - \beta_j) = 0$. The quantity

$$R_x[f, g] \equiv r^{n+1} u^{m+1} \prod_{i=1}^{m+1} \prod_{j=1}^{n+1} (\alpha_i - \beta_j) \quad (A-5)$$

is the resultant of the two polynomials f and g and is known to be equal to

$$\text{the determinant of the } (m+n+2) \times (m+n+2) \text{ matrix of Fig. 3.}$$

If the coefficients, r through w , are polynomials of y then $R_x[f, g]$ is a polynomial in y . The zeros of this polynomial are the values of y for which a zero of both functions (A-4) can be found. But we are focused on

$R_x[f, g](y)$ must be identically zero, since polynomials which are not identically zero only have isolated zeros.

The determinant of the matrix given in Fig. 3 is too complicated to be evaluated for all values of m and n , but fortunately when applied to the polynomials (A-3), the terms with the maximum and minimum powers of y , can be calculated. For any term in the determinant, the maximum power of y this term contributes to is $2(m+1)(n+1) - 2\delta_r - (n-1)\delta_u$ where δ_r and δ_u are the powers of r and u respectively. Thus, terms in the determinant which minimize $2\delta_r + (n-1)\delta_u$ are the terms which give the maximum power of y .

Similarly, the terms which contribute to the minimum power of y are the terms which minimize $2\delta_w + (m-1)\delta_t$ where δ_w and δ_t are the powers of w and t respectively. Because of the form of the polynomials (A-3) these terms are the same as those that give the maximum power of y .

Since each term in the determinant has one and only one element from each row and column of the matrix, it is easy to see that the minimum values of $2\delta_r + (n-1)\delta_u$ and $2\delta_w + (m-1)\delta_t$ are $2(n-1)$ and $2(m-1)$ respectively. The terms contributing to the determinant with these values are

$$\begin{aligned} & r^{n-1} t^2 v^{m+1} + r^{n-1} s^2 v^2 w^{m-1} + s^{n+1} u^2 w^{m-1} + s^{n-1} t^2 u^2 v^{m-1} \\ & + 2(-1)^{\frac{m-1}{2}} r^{n-1} s t v^{\frac{m+3}{2}} w^{\frac{m-1}{2}} + 2(-1)^{\frac{m-1}{2}} s^n t u^2 v^{\frac{m-1}{2}} w^{\frac{m-1}{2}} \\ & + 2(-1)^{\frac{n-1}{2}} r^{n-1} s^{\frac{n-1}{2}} t^2 u v^m + 2(-1)^{\frac{n-1}{2}} r^{\frac{n-1}{2}} s^{\frac{n+3}{2}} u v w^{\frac{m-1}{2}} \\ & + 4(-1)^{\frac{m-1}{2}} (-1)^{\frac{n-1}{2}} r^{\frac{n-1}{2}} s^{\frac{n+1}{2}} t u v^{\frac{m+1}{2}} w^{\frac{m-1}{2}} . \end{aligned} \quad (A-6)$$

For a given m and n , any term in this expression which has a fractional power of r through w is set equal to zero.

Substituting for r through w , the coefficients of the maximum or minimum power of y from the appropriate polynomials, the contribution to the resultant

for the largest and smallest powers in y is given by

$$R_x[f, g](y) = d^{m-1} (d-c)(d+(-1)^n c)(a-1)(c-1)(c+(-1)^n) y^{2m(n+1)+4} + \dots + b^{n-1} (b-a)(b+(-1)^m a)(c-1)(c+(-1)^n) y^{2m(n+1)+4} . \quad (A-7)$$

Similarly, because the polynomials (A-3) are of the same form when written as polynomials in y with coefficients that are polynomials in x , the largest and smallest powers of x in $R_y[f, g](x)$ can be calculated. Therefore,

$$R_y[f, g](x) = (-1)^{m+n} c^{m-1} (c-d)(c+(-1)^n d)(b-1)(b+(-1)^m b)(d-1)(d+(-1)^n) x^{2m-2} + \dots + (-1)^{m+n} a^{n-1} (a-b)(a+(-1)^m b)(d-1)(d+(-1)^n) x^{2m-2} \quad (A-8)$$

If a common factor of (A-3) depends on x and y , then both $R_y[f, g](x)$ and $R_x[f, g](y)$ must be identically zero. For m and n both odd, there are three disjoint possibilities:

$$\begin{aligned} (i) \quad a &= b = 1, \quad c \neq d \\ (ii) \quad c &= d = 1, \quad a \neq b \\ \text{and} \quad (iii) \quad a &= b \quad \text{and} \quad c = d . \end{aligned}$$

By comparing (i) with the original equations (A-2) one sees that it is spurious since (A-2a) implies that $\phi = \psi$ which, when substituted in (A-2b) implies that $c = d$. Case (ii) is similarly spurious. Case (iii) is the only possibility that corresponds to common factors. Obvious common factors are $(x-y)$ and $(x+y)$, and if $a = b = c = d = 1$ there are also $(x-1)$, $(x+1)$, $(y-1)$ and $(y+1)$. All these factors correspond to particles with equal rapidities or are not physically realizable. But there is still the possibility of other common factors depending on both x and y .

To exclude this possibility I use the transitivity property of the intersection matrix, S_{ij} , to rederive the polynomial equations, (A-3), with

particle '2' taken to be the reference particle. Proceeding as before the only possibility for an extra common factor is when $a = b = c = d = 1$, that is all particles have the same \hat{h}^{\pm} . After factoring out of the polynomials, $(\Lambda-3)$, the common factor $(x-y)(y-1)(x-1)$, the reduced polynomials are a sum of terms which all have the same sign. Thus, there can be no physically realizable extra common factor. Therefore, for m and n odd, there are only isolated solutions to $(\Lambda-1)$ when no two particles have the same rapidity.

For m and/or n even the possibilities (i) through (iii) are changed by the addition of minus signs. These extra minus signs are associated with two discrete symmetries of the original polynomial equations, $(A-3)$. If $x \rightarrow -x$, $a + (-1)^{m+1}a$ and $c + (-1)^{n+1}c$ or $y \rightarrow -y$, $b + (-1)^{m+1}b$ and $d + (-1)^{n+1}d$ then the polynomials remain invariant. By keeping track of these extra minus signs the same result is obtained as before, that is, that $(x-y)$, $(y-1)$ and $(x-1)$ are the only physically realizable common factors of $(A-3)$.

Thus for positive n there are only isolated solutions to $(A-3)$ and hence $(A-1)$ if no two particles have the same rapidity. This completes the proof of the lemma for positive n .

for negative n , factors which depend on both x and y can be obtained using the method outlined in this appendix. Unfortunately the resultants obtained from the polynomials $(\Lambda-3a)$ and $(\Lambda-3c)$ do not lend themselves to an easy evaluation of the maximum and minimum powers. However the results from an explicit algebraic calculation using MACSYMA for the cases $(m,n) = (3,-2)$,

$(4,-2)$, $(4,-3)$, $(5,-2)$, $(5,-3)$, and $(5,-4)$, show that the only physically realizable common factors are $(x-y)$, $(x-1)$ and $(y-1)$. I suspect this result is true for all negative n but do not know a general proof.

If this result is true in general, then there are only isolated solutions to $(A-1)$ if no two particles have the same rapidity and therefore the isolated points Lemma is true for negative as well as positive n .

Appendix B

In this appendix, I show that the maximum number of particles that have all pairs with the same intersection matrix, M_{ij} , is three times the number of ratios of \hat{h}^+/h^- in the mass spectrum. Two facts about the elements of the matrix M_{ij} are important. First, if a particle say i , is at rest, then $c_{ij} = -\hat{h}_i^+$ and $d_{ij} = -\hat{h}_i^-$ for all $j \neq i$. Second, under a Lorentz transformation

$$(a \pm c)_{ij} \rightarrow \Lambda^{m+n} (a \pm c)_{ij} \quad (B-1a)$$

$$\text{and } (b \pm d)_{ij} \rightarrow \Lambda^{m+n} (b \pm d)_{ij}. \quad (B-1b)$$

Hence, for a given intersection matrix M with elements a, b, c and d the Lorentz transformation parameterized by Λ , which takes us to a frame in which a particle is at rest, satisfies

$$\Lambda^{m+n} (a+c) - \Lambda^{m+n} (a-c) - \Lambda^2 (b+d) (\hat{h}^+/\hat{h}^-) + (b-d) (\hat{h}^+/\hat{h}^-) = 0. \quad (B-2)$$

Descartes' rule of sign tells us that the maximum number of positive solutions of a polynomial equation equals the number of sign changes of the coefficients. Hence, for every different (\hat{h}^+/\hat{h}^-) there are at most three different Lorentz transformations which bring a particle to rest. Since there are only massive particles in the spectrum, this is the required result.

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Footnote

- f1. To cover all possibilities, negative n satisfying $m > -n > 1$ should also be allowed. The statement and proof of the result is set up to deal with this case except that I do not have a general proof of the mathematical Lemma of Section III for such negative n even though I believe it to be true. See comments in Appendix A.

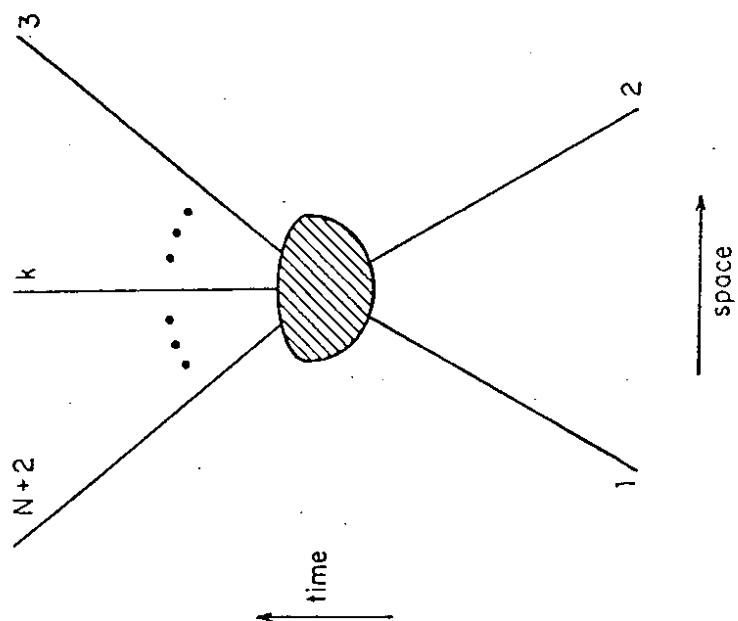


Figure 1. The spacetime diagram for a two-particle scattering process.

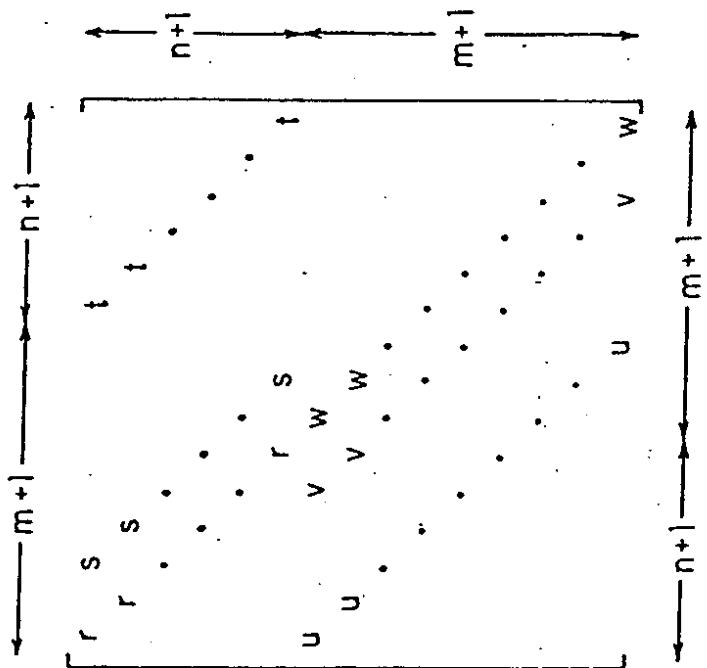


Figure 3. The matrix of coefficients.

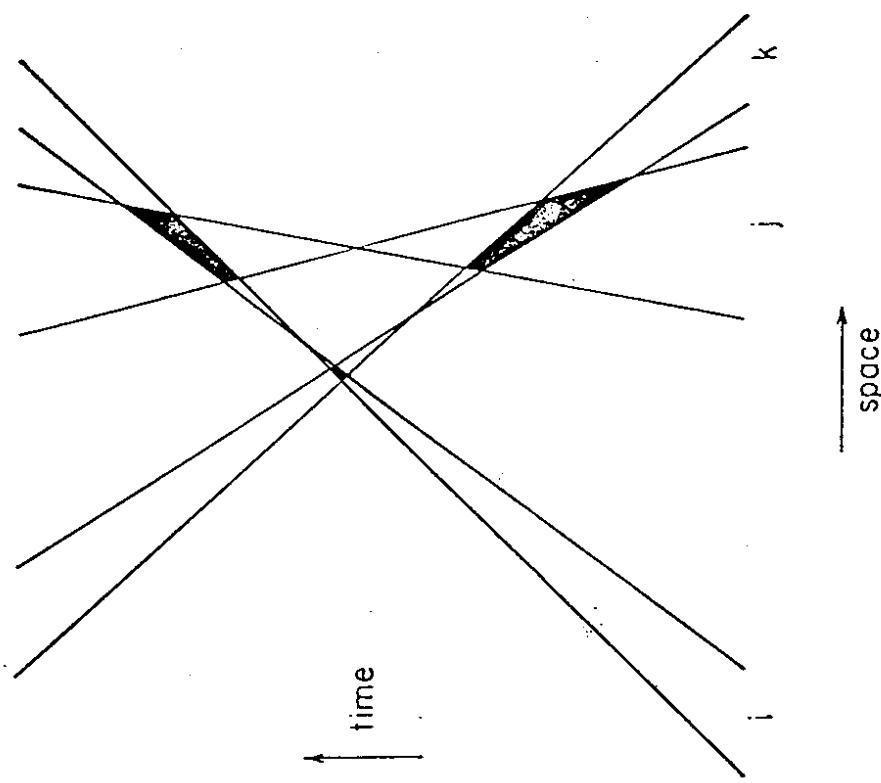


Figure 2. The spacetime regions of overlap for three wavepackets.